

INTEGRATION WITH RESPECT TO AN UPPER MEASURE FUNCTION

A Dissertation

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
THE UNIVERSITY OF MICHIGAN

BY

SHENG-CHIN FAN

1940

REPRINTED FROM THE AMERICAN JOURNAL OF MATHEMATICS,
Vol. LXIII, number 2, pages 319 to 338.



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Pierpont and Hildebrandt integrals are derived from theorem 11 and the additive properties of the latter integrals are obtained as an immediate consequence of theorems 2, 5. Because of the non-additivity and non-linearity and other peculiar properties (see remarks 4, 5), only a few theorems with special restrictions on the convergence of a sequence of integrals have been proved, and they form the material of part V.

Although we confine ourselves throughout the work to the Lebesgue upper measure of the set E , an examination of the proofs shows that all the results obtained remain valid for any upper measure function $\bar{\phi}(E)$ which satisfies the following axioms:

1. $\bar{\phi}(E) \geq 0$ for all sets E
2. $\bar{\phi}(E_1) \geq \bar{\phi}(E_2)$ for $E_1 \supseteq E_2$
3. $\bar{\phi}(E_1 + E_2) + \bar{\phi}(E_1 E_2) \leq \bar{\phi}(E_1) + \bar{\phi}(E_2)$
4. $\bar{\phi}(\sum_i E_i) \leq \sum_i \bar{\phi}(E_i)$ for any denumerable disjoint sets.

Furthermore, the sets E are assumed to be of finite upper measure and may be interpreted as the sets of n -dimensional euclidean space.

I. INTEGRALS OF μ -TYPE.

1. Definitions and preliminary remarks.

Definition 1. Let $f(x)$ be a bounded function defined on E . Let $\bar{\mu}(y) = \bar{m}E(f < y)$ be the upper measure of the set $E(f < y)$ and let $\sigma \equiv (m_f - \epsilon = y_0 < y_1 < y_2 < \dots < y_n = M_f + \epsilon)$ be a subdivision of the range of $f(x)$, where m_f, M_f are respectively the greatest lower bound (g. l. b.) and the least upper bound (l. u. b.) of $f(x)$ on E . Then if the limit, as the norm $|\sigma| = \max. (y_i - y_{i-1}) \rightarrow 0$, of the sum

$$(1) \quad S_\sigma = \sum_{i=1}^n \eta_i [\bar{\mu}(y_i) - \bar{\mu}(y_{i-1})], \quad y_{i-1} \leq \eta_i \leq y_i,$$

exists as a finite value, we say that the integral of $f(x)$ over E exists and denote this limiting value by $\int_E f d\bar{\mu}$.

Definition 2. By replacing in the above definition $\bar{\mu}(y_i) = \bar{m}E(f < y_i)$ by $\underline{\mu}(y_i) = \underline{m}E(f < y_i)$, the lower measure of the set $E(f < y_i)$, we define similarly the integral

$$(2) \quad \int_E f d\underline{\mu} = \lim_{|\sigma| \rightarrow 0} \sum_{i=1}^n \eta_i [\underline{\mu}(y_i) - \underline{\mu}(y_{i-1})]$$

Remark 1. The values of the integrals $\int_E f d\bar{\mu}$, $\int_E f d\underline{\mu}$ are unaltered,

if the sets $E(f < y_i)$ are replaced by the sets $E(f \leq y_i)$, $i = 1, 2, \dots, n$, in the above definitions. For let

$$\begin{aligned} S_n &= \sum_{i=1}^n y_i [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})] \\ s_n &= \sum_{i=1}^n y_{i-1} [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})] \\ S'_n &= \sum_{i=1}^n y_i [\bar{m}E(f \leq y) - \bar{m}E(f \leq y_{i-1})] \end{aligned}$$

then by rearranging the terms and by noting $\bar{m}E(f \leq y_0) = 0 = \bar{m}E(f < y_0)$ and $\bar{m}E(f \leq y_n) = \bar{m}E = \bar{m}E(f < y_n)$ we obtain without difficulty

$$\begin{aligned} S_n - S'_n &= \sum_{i=2}^n (y_i - y_{i-1}) [\bar{m}E(f \leq y_{i-1}) - \bar{m}E(f < y_{i-1})] \geq 0. \\ s_n - S'_n &= - \sum_{i=1}^n (y_i - y_{i-1}) [\bar{m}E(f < y_i) - \bar{m}E(f \leq y_{i-1})] \leq 0. \end{aligned}$$

Thus $s_n \leq S'_n \leq S_n$, and since both s_n , S_n approach $\int_E f d\bar{\mu}$, so also S'_n approaches $\int_E f d\bar{\mu}$.

Remark 2. The integrals $\int_E f d\bar{\mu}$, $\int_E f d\mu$ exist for all bounded functions $f(x)$ defined on E .

This follows immediately from the existence of a Stieltjes integral $\int g d\alpha$ where g is continuous and α is of bounded variation. Now $\int_E f d\bar{\mu}$ is essentially a Stieltjes integral, i. e.

$$(3) \quad \int_E f d\bar{\mu} = \int_A^B y d\bar{m}E(f < y), \quad A < f < B,$$

so that we have here $g = y$ and $\alpha = \bar{\mu}$ where $\bar{\mu}(y)$ is a monotonically non-decreasing function of y .

Remark 3. The functions $\bar{\mu}(y) \equiv \bar{m}E(f < y)$ and $\tilde{\mu}(y) \equiv \bar{m}E(f \leq y)$ are equal except possibly for a denumerable set of points of discontinuity of the function $\tilde{\mu} - \bar{\mu}$.

By applying the integration by parts theorem [1, p. 28] to the corresponding Stieltjes integral in (3) we get

$$(4) \quad \int_E f d\bar{\mu} = \int_A^B y d\bar{m}E(f < y) = B\bar{m}E - \int_A^B \bar{m}E(f < y) dy,$$

and similarly

$$(5) \quad \int_E f d\tilde{\mu} = \int_A^B y d\bar{m}E(f \leq y) = B\bar{m}E - \int_A^B \bar{m}E(f \leq y) dy.$$

$$\begin{aligned} \bar{m}(E_1 + E_2 + \cdots + E_{2p}) + \bar{m}(E_{2p+1} + E_{2p+2}) - \bar{m}(E_1 + E_2 + \cdots + E_{2p+2}) \\ \geq \bar{m}(E_2 + E_4 + \cdots + E_{2p}) + \bar{m}E_{2p+2} - \bar{m}(E_2 + E_4 + \cdots + E_{2p+2}) \end{aligned}$$

and adding this inequality to (7), we see that it is true $2p + 2$ sets.

THEOREM 3. If $E_1, E_2, \cdots$ constitute an arbitrary division of E , then

$$(9) \quad f \geq 0, \quad \int_E f d\bar{\mu} \leq \int_{E_1} f d\bar{\mu} + \int_{E_2} f d\bar{\mu} + \cdots,$$

$$(10) \quad f \leq 0, \quad \int_E f d\bar{\mu} \geq \int_{E_1} f d\bar{\mu} + \int_{E_2} f d\bar{\mu} + \cdots.$$

Proof. To prove (9), suppose there are p disjoint subsets of E . Then (7) gives immediately the inequality

$$\begin{aligned} \bar{m}E_1 + \bar{m}E_2 + \cdots + \bar{m}E_p - \bar{m}E \geq \bar{m}E_1(f < y) \\ + \bar{m}E_2(f < y) + \cdots + \bar{m}E_p(f < y) - \bar{m}E(f < y). \end{aligned}$$

Integrating both sides in the Riemann sense from 0 to B , where $0 \leq f < B$, we have

$$\begin{aligned} B(\bar{m}E_1 + \bar{m}E_2 + \cdots + \bar{m}E_p - \bar{m}E) \geq \int_0^B [\bar{m}E_1(f < y) \\ + \bar{m}E_2(f < y) + \cdots + \bar{m}E_p(f < y) - \bar{m}E(f < y)] dy, \end{aligned}$$

or

$$\begin{aligned} B\bar{m}E - \int_0^B \bar{m}E(f < y) dy \leq B(\bar{m}E_1 + \bar{m}E_2 + \cdots + \bar{m}E_p) \\ - \int_0^B [\bar{m}E_1(f < y) + \bar{m}E_2(f < y) + \cdots + \bar{m}E_p(f < y)] dy, \end{aligned}$$

that is by means of (4),

$$\int_E f d\bar{\mu} \leq \int_{E_1} f d\bar{\mu} + \int_{E_2} f d\bar{\mu} + \cdots + \int_{E_p} f d\bar{\mu}.$$

Next, for the infinite case, denoting the sum of the first p sets by $E^{(p)}$ then we have just proved

$$\int_A^B y d\bar{m}E^{(p)}(f < y) = \int_{E^{(p)}} f d\bar{\mu} \leq \int_{E_1} f d\bar{\mu} + \int_{E_2} f d\bar{\mu} + \cdots + \int_{E_p} f d\bar{\mu}.$$

By taking the limits of both sides, we obtain

$$\lim_{p \rightarrow \infty} \int_A^B y d\bar{m}E^{(p)}(f < y) \leq \int_{E_1} f d\bar{\mu} + \int_{E_2} f d\bar{\mu} + \cdots.$$

Now since the functions $\bar{m}E^{(p)}(f < y)$ are uniformly of bounded variation and $\bar{m}E^{(p)}(f < y) \rightarrow \bar{m}E(f < y)$ for each y , then by a theorem on the Stieltjes integral [2, p. 180; 7, p. 272], we conclude that

$$\lim_{p \rightarrow \infty} \int_A^B y d\tilde{m}E^{(p)}(f < y) = \int_A^B y d\tilde{m}E(f < y),$$

and consequently (9) is proved.

Lastly for (10), it reduces to proving that for $A \leq f \leq 0$,

$$\int_A^0 \tilde{m}E(f < y) dy \leq \int_A^0 [\tilde{m}E_1(f < y) + mE_2(f < y) + \cdots] dy,$$

and this is evidently true.

Remark 4. Since the integral is not additive, we can not expect that the inequality $\int_E f d\bar{\mu} \geq \int_{E_1} f d\bar{\mu}$ holds for non-negative f and any subset E_1 of E . Indeed, the following example illustrates this case. Let $E = E_1 + E_2$, $f(x)$ be the characteristic function of E_1 , then

$$\int_E f d\bar{\mu} = \tilde{m}(E_1 + E_2) - \tilde{m}E_2, \quad \int_{E_1} f d\bar{\mu} = \tilde{m}E_1$$

and so

$$\int_E f d\bar{\mu} \leq \int_{E_1} f d\bar{\mu}.$$

4. Non-Linearity. Before we consider in theorem 6 the non-linearity of the integral $\int_E (f + g) d\bar{\mu}$, we shall take up two special cases in which one of the functions f, g is a constant and one of them is relatively measurable (r.m.) on E . These are given respectively by the following two theorems.

THEOREM 4. *If c is a constant, then*

$$\int_E (f + c) d\bar{\mu} = \int_E f d\bar{\mu} + c\tilde{m}E.$$

THEOREM 5. *If one of the bounded functions f, g is r.m., then*

$$(11) \quad \int_E (f + g) d\bar{\mu} = \int_E f d\bar{\mu} + \int_E g d\bar{\mu}.$$

Proof of THEOREM 4. Let $f(x)$ be bounded by (A, B) , then $f + c$ is bounded by $(A + c, B + c)$. The theorem follows easily from formula (4) and

$$\begin{aligned} \int_E (f + c) d\bar{\mu} &= (B + c)\tilde{m}E - \int_{A+c}^{B+c} \tilde{m}E(f + c < y) dy \\ &= (B + c)\tilde{m}E - \int_A^B \tilde{m}E(f < y) dy. \end{aligned}$$

Proof of THEOREM 5. Suppose g is r.m., and divide the range of g by ($m_g = y_0 < y_1 < y_2 \cdots < y_n = M_g$). Let $E_i = E(y_{i-1} \leq g < y_i)$; then $E_1, E_2, \cdots, E_n$ constitute a separated division of E [6, p. 127], and

$$\begin{aligned} \int_E (f+g) d\bar{\mu} &= \sum_{i=1}^n \int_{E_i} (f+g) d\bar{\mu}, \text{ by Theorem 2} \\ &\leq \sum_{i=1}^n \int_{E_i} (f+y_i) d\bar{\mu}, \text{ by Theorem 1(e)} \\ &= \int_E f d\bar{\mu} + \sum_{i=1}^n y_i \bar{m} E_i, \text{ by Theorems 4, 2.} \end{aligned}$$

On the other hand, in the same way,

$$\int_E (f+g) d\bar{\mu} \geq \int_E f d\bar{\mu} + \sum_{i=1}^n y_{i-1} \bar{m} E_i.$$

Since both summations in the above two inequalities approach the same limit $\int_E g d\bar{\mu}$, the equality (11) is established.

THEOREM 6. *If f, g are any two bounded functions on E , then*

$$(12) \quad \int_E (f+g) d\bar{\mu} \geq \int_E f d\bar{\mu} + \int_E g d\bar{\mu}.$$

Proof. Without loss of generality, we assume that the functions f, g are positive since all the other cases may be reduced to this case by applying Theorem 4. Let both f and g be bounded by ($0 < A, B$). Then $f+g$ is bounded by ($2A, 2B$). Apply formula (4) respectively to the integrands $f+g, f$ and g ; then the proof of (12) is reduced to the proof of the inequality between Riemann integrals

$$(13) \quad \int_A^B [\bar{m}E(f < y) + \bar{m}E(g < y)] dy \geq \int_{2A}^{2B} \bar{m}E(f+g < y) dy.$$

For this latter purpose, divide the range (A, B) of f, g into n equal parts and the range ($2A, 2B$) of $f+g$ into $2n$ equal parts each of length $l = \frac{B-A}{n}$. Write

$$\begin{aligned} E_i &= E(A + (i-1)l \leq f < A + il) & i &= 1, 2, \cdots, n \\ E'_j &= E(A + (j-1)l \leq g < A + jl) & j &= 1, 2, \cdots, n \\ e_k &= E(2A + (k-1)l \leq f+g < 2A + kl) & k &= 1, 2, \cdots, 2n, \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_1 &= E_1 E'_1, \mathcal{E}_2 = E_1 E'_2 + E_2 E'_1, \cdots \mathcal{E}_n = E_1 E'_n + E_2 E'_{n-1} + \cdots + E_n E'_1 \\ \mathcal{E}_{n+1} &= E_2 E'_n + E_3 E'_{n-1} + \cdots + E_n E'_2, \\ &\cdots \mathcal{E}_{2n-2} = E_{n-1} E'_n + E_n E'_{n-1}, \mathcal{E}_{2n-1} = E_n E'_n, \end{aligned}$$

is $(E_1 + E_2) E_1' + E_1 (E_1' + E_2') = E_1 E_1' + (E_1 E_2' + E_2 E_1') = \mathcal{E}_1 + \mathcal{E}_2$
and in general the p -th term ($1 \leq p \leq n$) is

$$\begin{aligned} & (E_1 + E_2 + \cdots + E_p) E_1' + (E_1 + E_2 + \cdots + E_{p-1}) (E_1' + E_2') + \\ & \quad + \cdots + (E_1 + E_2) (E_1' + E_2' + \cdots + E_{p-1}') \\ & \quad + E_1 (E_1' + E_2' + \cdots + E_p') \\ & = E_1 E_1' + (E_1 E_2' + E_2 E_1') + \cdots + (E_1 E_p' + E_2 E_{p-1}' + \cdots + E_p E_1') \\ & = \sum_{i=1}^p \mathcal{E}_i, \end{aligned}$$

and the $(2n - q - 1)$ -th term ($1 \leq q \leq n - 2$) is

$$\begin{aligned} & \sum_1^{n-q} E_i \sum_1^n E_i' + \sum_1^{n-q+1} E_i \sum_1^{n-1} E_i' + \sum_1^{n-q+2} E_i \sum_1^{n-2} E_i' + \cdots + \\ & \quad + \sum_1^{n-1} E_i \sum_1^{n-q+1} E_i' + \sum_1^n E_i \sum_1^{n-q} E_i' \\ & = E_1 E_1' + (E_1 E_2' + E_2 E_1') + \cdots + (E_1 E_n' + \cdots + E_n E_1') \\ & \quad + (E_2 E_n' + \cdots + E_n E_2') + \cdots + (E_{n-q} E_n' + E_{n-q+1} E_{n-1}' + \cdots + E_n E_{n-q}') \\ & = \sum_{i=1}^{2n-q-1} \mathcal{E}_i. \end{aligned}$$

This proves the lemma.

COROLLARY.
$$\int_E (f - g) d\bar{\mu} \leq \int_E f d\bar{\mu} - \int_E g d\bar{\mu}.$$

II. INTEGRALS OF THE μ^* -TYPE.

5. Relations between integrals of μ -type and μ^* -type. So far we have developed the properties of the integrals of μ -type defined by Definitions 1, 2. If we replace in these two definitions $\bar{\mu}(y) = \bar{m}E(f < y)$ by $\mu^*(y) = \bar{m}E(f > y)$ and $\underline{\mu}(y) = \underline{m}E(f < y)$ by $\mu^*(y) = \underline{m}E(f > y)$ respectively, we have two μ^* -type integrals. Thus

Definition 3.

$$\int_E f d\bar{\mu}^* = \lim_{|\sigma| \rightarrow 0} S_{\sigma}^* = \lim_{|\sigma| \rightarrow 0} \sum_{i=1}^n \eta_i [\bar{m}E(f > y_{i-1}) - \bar{m}E(f > y_i)].$$

Definition 4.

$$\int_E f d\underline{\mu}^* = \lim_{|\sigma| \rightarrow 0} s^* = \lim_{|\sigma| \rightarrow 0} \sum_{i=1}^n \eta_i [\underline{m}E(f > y_{i-1}) - \underline{m}E(f > y_i)].$$

We note that the remarks 1, 2, 3 apply here as well except that the formula (4) takes the form

$$(16) \quad \int_E f d\bar{\mu}^* = - \int_A^B y d\bar{m}E(f > y) = A\bar{m}E + \int_A^B \bar{m}E(f > y) dy.$$

THEOREM 7.
$$\int_E f d\bar{\mu}^* = - \int_E (-f) d\bar{\mu}.$$

Proof. The theorem follows at once from the relation

$$\begin{aligned} \int_E (-f) d\bar{\mu} &= \int_{-B}^{-A} y d\bar{\mu} = -A\bar{m}E - \int_{-B}^{-A} \bar{m}E(-f < y) dy \\ &= -A\bar{m}E - \int_A^B \bar{m}E(f > y) dy \end{aligned}$$

and the formula (16).

By means of this theorem, all properties which are expressed in theorems 1-6 can be easily transferred to the integrals $\int f d\bar{\mu}^*$.

THEOREM 8.
$$\int_E f d\bar{\mu} \leq \int_E f d\bar{\mu}^*.$$

Proof. By the formulas (4), (16) and remark 1, we need to prove only

$$(17) \quad \int_A^B [\bar{m}E(f \leq y) + \bar{m}E(f > y) - \bar{m}E] dy \geq 0,$$

and this is evidently true, since the integrand is never negative.

Remark 5. For $f \leq 0$, by means of Theorems 7, 8,

$$\left| \int f d\bar{\mu} \right| = - \int f d\bar{\mu} \geq \int (-f) d\bar{\mu} = \int |f| d\bar{\mu}.$$

For f of an arbitrary sign, there exists no such relation as is shown by the following examples:

1° Let E_1, E_2 be any two of the denumerable non-measurable Vitali sets [4, p. 575; 5, p. 401] such that $E = E_1 + E_2$, $\bar{m}E_1 = \bar{m}E_2 > 0$. Define f to be 1 on E_1 , -1 on E_2 ; then

$$\left| \int_E f d\bar{\mu} \right| = 2\bar{m}E_2 - \bar{m}E \leq \bar{m}E = \int_E |f| d\bar{\mu}.$$

2° On the other hand, let E be the set of irrational points in $(0, 1)$, E_1, E_2 be the Van Vleck sets [16] such that $\bar{m}E = \bar{m}E_1 = \bar{m}E_2 = 1$. Define f to be 1 on E_1 , - n on E_2 , then

$$\left| \int_E f d\bar{\mu} \right| = n > 1 = \int_E |f| d\bar{\mu}.$$

6. Non-Linearity. As a complement to the non-linearity property given in theorem 6, we prove in this section two more theorems.

THEOREM 9. *If both f and g are bounded, then*

$$(18) \quad \int f d\bar{\mu} + \int g d\bar{\mu} \leq \int (f + g) d\bar{\mu} \leq \int f d\bar{\mu} \\ + \int g d\bar{\mu}^* \leq \int (f + g) d\bar{\mu}^* \leq \int f d\bar{\mu}^* + \int g d\bar{\mu}^*.$$

Proof. The first inequality is a restatement of Theorem 6, the last is an immediate consequence of Theorems 6, 7, and the middle can be established by means of the corollary to Theorem 6 and Theorem 7.

THEOREM 10. *If c is a constant, then*

$$(19) \quad \text{for } c \geq 0: \int c f d\bar{\mu} = c \int f d\bar{\mu},$$

$$(20) \quad \text{for } c \leq 0: \int c f d\bar{\mu} \leq c \int f d\bar{\mu}.$$

Proof. For $c \geq 0$, the equality follows easily from the definition by the usual method. For $c \leq 0$,

$$\begin{aligned} \int c f d\bar{\mu} &= \int (-c)(-f) d\bar{\mu} = (-c) \int (-f) d\bar{\mu} \\ &= c(-\int (-f) d\bar{\mu}) \\ &= c \int f d\bar{\mu}^* \text{ by Theorem 7} \\ &\leq c \int f d\bar{\mu} \text{ by Theorem 8.} \end{aligned}$$

7. A condition for μ -integrability Define f to be μ -integrable in case $\int_E f d\bar{\mu} = \int_E f d\bar{\mu}^*$. Then

THEOREM 11. *In order that a bounded function f be μ -integrable over E , it is necessary and sufficient that the function f should be relatively measurable on E .*

Proof. The sufficiency is obvious. To prove the necessity, we derive, as we have done for (17),

$$\int_A^B [\bar{m}E(f \leq y) + \bar{m}E(f > y) - \bar{m}E] dy = 0.$$

From the monotonicity of the functions $\bar{m}E(f \leq y)$, $\bar{m}E(f > y)$ it follows that the integrand must vanish except at most for a denumerable number of points. This suffices to insure the relative measurability of f on E [14, p. 12].

COROLLARY 1. *If f is μ -integrable on E , then it is μ -integrable on any subset of E .*

COROLLARY 2. If f is $\bar{\mu}$ -integrable on E , then both f^+ , f^- are $\bar{\mu}$ -integrable and conversely.

COROLLARY 3. If f is $\bar{\mu}$ -integrable, so is $|f|$, but the converse does not always hold.

III. INTEGRALS OF UNBOUNDED FUNCTIONS.

8. μ -summable functions.

Definition 5. If f is unbounded on E , divide the whole y -axis by a subdivision $\sigma \equiv (\cdots y_{-n} < \cdots y_{-1} < y_0 = 0 < y_1 < y_2 \cdots < y_n \cdots)$ and define the integral

$$(21) \quad \int_E f d\bar{\mu} = \lim_{|\sigma| \rightarrow 0} S_\sigma = \lim_{|\sigma| \rightarrow 0} \sum_{i=-\infty}^{\infty} \eta_i [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})],$$

when the limit exists as a finite value.

Similarly, we may define the integrals $\int f d\mu$, $\int f d\mu^*$, $\int f d\mu^*$.

Definition 6. If there exists a subdivision σ_0 such that the series

$$(22) \quad S_{\sigma_0} = \sum_{i=-\infty}^{\infty} y_i [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})]$$

is convergent, then f is $\bar{\mu}$ -summable.

Similar definitions for μ , μ^* , μ^* -summable.

THEOREM 12. If f is $\bar{\mu}$ -summable, then the integral $\int_E f d\bar{\mu}$ exists.

Proof. In the first place, we note that if (22) is convergent, it is absolutely convergent, i. e. the series of positive terms and the series of negative terms in (22) are both convergent. Next, for any subdivision $\sigma \geq \sigma_0$, i. e. σ is finer than σ_0 , $|S_\sigma - S_{\sigma_0}| \leq |\sigma_0| \bar{m}E$. Thirdly, S_σ is also convergent. For the series of positive terms and the series of negative terms in S_σ can be shown to be convergent. Lastly, for $e > 0$ let σ_1, σ_2 be such that $\sigma_1 \geq \sigma_0$, $\sigma_2 \geq \sigma_0$ and the norms $|\sigma_1|$, $|\sigma_2|$ be not greater than $e/2\bar{m}E$, then the absolute value of the difference of the sums

$$\begin{aligned} |S_{\sigma_1} - S_{\sigma_2}| &\leq |S_{\sigma_1} - S_{\sigma_1\sigma_2}| + |S_{\sigma_1\sigma_2} - S_{\sigma_2}| \\ &\leq |\sigma_1| \bar{m}E + |\sigma_2| \bar{m}E \leq e/2 + e/2 = e, \end{aligned}$$

and this proves the theorem as e is arbitrary.

THEOREM 13. If we define the integral as the limit

$$(23) \quad \int_E f d\bar{\mu} = \lim_{N \rightarrow \infty} \int_{-N}^N y d\bar{m}E(f < y),$$

then this is equivalent to definition 5.

Proof. 1°: suppose the integral exists in the sense of (21), or, what amounts to the same thing, that it is the limit of

$$\sum_{i=1}^{\infty} y_i [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})] + \sum_{i=-\infty}^0 y_{i-1} [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})].$$

Then corresponding to $\epsilon > 0$, there exists a number N such that $y_k = N$, $y_{-k-1} = -N$ and the remainder R_N in the first summation and the remainder R_{-N} in the second summation are both less than ϵ in absolute value. So we may write on denoting η_i by y_i or y_{i-1} according as it is positive or negative,

$$\begin{aligned} \int_E f d\bar{\mu} &= \lim_{|\sigma| \rightarrow 0} \left[\sum_{i=-k'}^k \eta_i [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})] + R_N + R_{-N} \right] \\ &= \int_{-N}^N y d\bar{m}E(f < y) + \lim_{|\sigma| \rightarrow 0} (R_N + R_{-N}) \end{aligned}$$

But in taking the limit of the last term it is permissible to use only consecutive subdivisions, therefore

$$\left| \int_E f d\bar{\mu} - \int_{-N}^N y d\bar{m}E(f < y) \right| \leq 2\epsilon,$$

and this gives (23).

2° Suppose the integral exists in the sense of (23) or as the sum of the limits $\lim_{N \rightarrow \infty} \int_{-N}^0 y d\bar{m}E(f < y) dy$ and $\lim_{N \rightarrow \infty} \int_0^N y d\bar{m}E(f < y)$. Let $N = y_k$ and n be the number of points of division of the range $(0, N)$ (Here we use consecutive subdivisions with the norm approaching zero), then

$$\lim_{N \rightarrow \infty} \int_0^N y d\bar{m}E(f < y) = \lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^k y_{i-1} [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})].$$

Now the sum on the right may be considered as a function of two variables N, n and it is non-decreasing as both N, n increase. On putting $\alpha = 1/N$, $\beta = 1/n$, this sum is a monotonic function $F(\alpha, \beta)$ of α, β such that $\lim_{\alpha \rightarrow 0} F(\alpha, \beta) = \lim_{\beta \rightarrow 0} \lim_{\alpha \rightarrow 0} F(\alpha, \beta)$ [8, Vol. I. § 307]. Hence

$$(24) \quad \lim_{N \rightarrow \infty} \int_0^N y d\bar{m}E(f < y) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} y_{i-1} [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})],$$

and similarly

$$(25) \quad \lim_{N \rightarrow \infty} \int_{-N}^0 y d\bar{m}E(f < y) = \lim_{n \rightarrow \infty} \sum_{i=-\infty}^0 y_i [\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})].$$

That is, the integral exists in the sense of definition (5).

COROLLARY. *If the integral $\int_E f d\bar{\mu}$ exists, then*

$$N[\bar{m}E - \bar{m}E(f < N)] \rightarrow 0, \quad N\bar{m}E(f < -N) \rightarrow 0.$$

Proof. $N[\bar{m}E - \bar{m}(f < N)] \leq \sum_{i=k+1}^{\infty} y_{i-1}[\bar{m}E(f < y_i) - \bar{m}E(f < y_{i-1})]$,

where $y_k = N$. The summation on the right being a remainder of a convergent series, approaches zero as $N \rightarrow \infty$. Similarly, $N\bar{m}E(f < -N) \rightarrow 0$.

9. Properties deduced from integrals of bounded functions. Making use of Theorems 12, 13 and the important corollary to Theorem 13, we can extend the properties already obtained from Theorem 1 to Theorem 11 for bounded functions to unbounded functions. In the following we merely state the corresponding theorems for $\bar{\mu}$ -type integral and omit the proofs.

THEOREM 1. (a), (b), (c) still hold for $\bar{\mu}$ -summable functions. (d) becomes: if f is $\bar{\mu}$ -summable, then f^+ , f^- are $\bar{\mu}$ -summable and $\int f d\bar{\mu} = \int f^+ d\bar{\mu} + \int f^- d\bar{\mu}$. Also conversely. For (e), if $f \geq g$ are both $\bar{\mu}$ -summable, then $\int f d\bar{\mu} \geq \int g d\bar{\mu}$.

THEOREM 2. If f is $\bar{\mu}$ -summable on $E_1, E_2, \dots$ of a separated division of E , then it is $\bar{\mu}$ -summable on E and the equality (6) holds, and conversely.

THEOREM 3. If f is $\bar{\mu}$ -summable on E , then (9), (10) are true.

THEOREM 4. If f is $\bar{\mu}$ -summable, then $f + c$ is $\bar{\mu}$ -summable and the same relation holds.

THEOREMS 5, 6 hold for f, g and $f + g$ are $\bar{\mu}$ -summable.

THEOREM 7. If $(-f)$ is $\bar{\mu}$ -summable, then f is $\bar{\mu}^*$ -summable and $\int f d\bar{\mu}^* = - \int (-f) d\bar{\mu}$ and conversely.

THEOREMS 8, 9 are still true if the functions involved are both $\bar{\mu}$ and $\bar{\mu}^*$ -summable.

THEOREM 10. If f is $\bar{\mu}$ -summable, then for $c \geq 0$, cf is $\bar{\mu}$ -summable and $\int cf d\bar{\mu} = c \int f d\bar{\mu}$. If f is also $\bar{\mu}^*$ -summable then for $c < 0$, cf is $\bar{\mu}$ -summable and $\int cf d\bar{\mu} \leq c \int f d\bar{\mu}$.

Theorem 11 is still true if f is $\bar{\mu}$ -summable.

IV. RELATIONS TO PIERPONT AND HILDEBRANDT INTEGRALS.

10. Relations. Jeffrey has defined the Hildebrandt integral $II \int_E f dx$ as the limit in the norm sense of the sum $\sum_{i=-\infty}^{\infty} \eta_i \bar{m} E_i$ corresponding to a subdivision σ . Merely replacing the upper $\bar{m} E_i$ by lower measure $m E_i$, we define the Hildebrandt $H_0 \int_E f dx$. For the definition of Pierpont P -integral and P_0 -integral, we refer to the Pierpont book [13, p. 372] and the Hildebrandt paper [6, p. 178]. In view of Theorem 11, we get readily the relations between integrals expressed by

THEOREM 14. *If f is μ -summable (upper or lower) and μ -integrable (upper or lower) in the sense of § 7, then it is P , H , P_0 and H_0 -integrable and*

$$(26) \quad \int_E f d\bar{\mu} = \int_E f d\bar{\mu}^* = H \int_E f dx = P \int_E f dx$$

$$(27) \quad \int_E f d\mu = \int_E f d\mu^* = H_0 \int_E f dx = P_0 \int_E f dx.$$

Since the existence of $P \int f$ involves necessarily the relative measurability of f , while $P_0 \int f$, [6, p. 127], $II \int f$, $H_0 \int f$, [9, Theorem 5], may exist for non-relatively measurable functions, then the converse of Theorem 14 may be formulated as the following

THEOREM 15.

1° *If f is P -integrable, then it is $\bar{\mu}$, $\bar{\mu}^*$ -summable and $\bar{\mu}$ -integrable and*

$$\int f d\bar{\mu} = \int f d\bar{\mu}^* = P \int f dx,$$

2° *If f is H -integrable, and f is not r. m., then f is $\bar{\mu}$, $\bar{\mu}^*$ -summable and*

$$\text{for } f \geq 0 \quad 0 \leq \int f d\bar{\mu} < \int f d\bar{\mu}^* \leq H \int f dx$$

$$\text{for } f \leq 0 \quad H \int f dx \leq \int f d\bar{\mu} < \int f d\bar{\mu}^* \leq 0,$$

3° *If f is H_0 -integrable, and f is not r. m., then*

$$\text{for } f \geq 0 \quad 0 \leq H_0 \int f \leq \int f d\mu^* < \int f d\mu$$

$$\text{for } f \leq 0 \quad \int f d\mu^* < \int f d\mu \leq H_0 \int f dx \leq 0.$$

The only thing which needs proof is the first part of 2°. Since the integral $H \int f dx$ exists, both the integrals $H \int f^+ dx$, $H \int f^- dx$ exist. From the relations in the second part of 2°, it is easily seen that the functions f^+ , f^- are both $\bar{\mu}$, $\bar{\mu}^*$ -summable and by Theorem 1 (d) of § 9, we conclude that f is $\bar{\mu}$, $\bar{\mu}^*$ -summable.

11. Some elementary properties of Pierpont and Hildebrandt integrals for r. m. functions may be deduced from the properties obtained for the integrals of μ -type. For example, we have from Theorem 2 and Theorem 5 the additive properties:

$$\begin{aligned} \int_E f dx &= \int_{E_1} f dx + \int_{E_2} f dx + \cdots, \\ \int_E (f_1 + f_2 + \cdots + f_p) dx &= \int_E f_1 dx + \int_E f_2 dx + \cdots + \int_E f_p dx, \end{aligned}$$

where f , f_1 , f_2 , $\cdots$, f_p are r. m. on E and E_1 , E_2 , $\cdots$ constitute a separated division of E and the integrals are either P , P_0 , H or H_0 integrable throughout.

Before we consider in the next part the convergence of a sequence of integrals of μ -type, let us add here a theorem of this kind for Hildebrandt integrals: If we are given a sequence of r. m. functions f_n which are uniformly bounded and $f_n \rightarrow f$ on E , then f is r. m. and $H \int_E f dx = \lim_n H \int_E f_n dx$. The proof rests essentially on the fact that $\bar{m}E(|f - f_n| > \eta) \rightarrow 0$ for any positive η and this can be established exactly as for measurable f_n [10, p. 105].

V. CONVERGENCE OF SEQUENCES OF INTEGRALS.

12. Monotonic Sequences of functions.

THEOREM 16. *If $\{f_n\}$ is a monotonically non-increasing sequence of bounded functions on E and if $f_n \rightarrow f$ which is also bounded, then*

$$(28) \quad \int_E f d\bar{\mu} = \lim_n \int_E f_n d\bar{\mu}.$$

Proof. Since the sequence $\{f_n\}$ is monotonically non-increasing with $f_n \rightarrow f$, we have for each y a sequence of sets $\{E(f_n < y)\}$ each containing the preceding with the limiting set $E(f < y) = \bigcap_{n=1}^{\infty} E(f_n < y)$, and by a theorem of Young [17, Theorem 28, p. 104], $\bar{m}E(f_n < y) \rightarrow \bar{m}E(f < y)$. Moreover, the functions $\bar{m}E(f < y)$ of y are non-decreasing and of uniformly

bounded variation with total variation equal to $\bar{m}E$. Hence by a theorem on a sequence of Stieltjes integrals [2, p. 180; 7, p. 272], we have at once

$$\int_A^B y d\bar{m}E(f < y) = \lim_n \int_A^B y d\bar{m}E(f_n < y).$$

This is (28) and the theorem is proved.

In like manner we prove

THEOREM 17. *If $\{f_n\}$ is a monotonically non-decreasing sequence of bounded functions on E and if $f_n \rightarrow f$ which is also bounded, then*

$$(29) \quad \int_E f d\mu = \lim_n \int_E f_n d\mu.$$

The above two theorems are still true, if the word bounded is replaced by the word μ -summable ($\bar{\mu}$, μ respectively). But an example can be constructed (see example in Theorem 20) to show that (28) is not always true for a monotonically non-decreasing sequence $\{f_n\}$ while (29) is not always true for a monotonically non-increasing sequence.

13. Other Sequences of functions. With the help of Theorems 16, 17 we can derive the following

THEOREM 18. *If $\{f_n\}$ is a sequence of uniformly bounded functions, then*

$$(30) \quad \int_E \overline{\lim} f_n d\bar{\mu} \geq \overline{\lim}_n \int_E f_n d\bar{\mu},$$

$$(31) \quad \int_E \underline{\lim} f_n d\mu \leq \underline{\lim}_n \int_E f_n d\mu.$$

Proof. The existence of the integral $\int_E \overline{\lim} f_n d\bar{\mu}$ follows from the necessary boundedness of the integrand. Let $h_n(x) = \text{l. u. b. } [f_n(x), f_{n+1}(x), \dots]$ for every x of E , then $\{h_n\}$ is a monotonically non-increasing sequence of bounded functions such that $h_n \geq f_n$ and $\lim h_n = \lim f_n$. Hence by Theorem 16 and Theorem 1(e),

$$\int_E \overline{\lim} f_n d\bar{\mu} = \lim \int_E h_n d\bar{\mu} \geq \overline{\lim} \int_E f_n d\bar{\mu}.$$

Similarly for (31).

THEOREM 19. *If $\{f_n\}$ is a sequence of uniformly bounded functions and*

f_n approaches a bounded function f in a measure in the sense that $\bar{m}E(|f_n - f| > \eta) \rightarrow 0$ for all positive η and if $f - f_n$ is of constant sign on E , then

$$\int_E f d\bar{\mu} = \lim_n \int_E f_n d\bar{\mu}.$$

Proof. Suppose $f - f_n \geq 0$ and let $E_{n\eta} = E(f - f_n > \eta)$, then by means of the corollary to Theorem 6 and Theorems 7, 3 1(a), we have

$$\begin{aligned} 0 &\leq \int_E f d\bar{\mu} - \int_E f_n d\bar{\mu} \leq - \int_E (f_n - f) d\bar{\mu} = \int_E (f - f_n) d\bar{\mu}^* \\ &\leq \int_{E_{n\eta}} (f - f_n) d\bar{\mu}^* + \int_{E - E_{n\eta}} (f - f_n) d\bar{\mu}^* \\ &\leq 2M\bar{m}E_{n\eta} + \eta\bar{m}(E - E_{n\eta}) \end{aligned}$$

which approaches zero since η is arbitrary.

THEOREM 20. If $\{f_n\}$ is a sequence of functions such that $f_n \rightarrow f$ in a measure, then

$$(32) \quad \lim_n \int_E \frac{|f_n - f|}{1 + |f_n - f|} d\bar{\mu} = 0.$$

Proof. Let $E_{n\eta} = E(|f_n - f| > \eta)$, $E'_{n\eta} = E\left(\frac{|f_n - f|}{1 + |f_n - f|} > \eta\right)$ then $E_{n\eta} \supseteq E'_{n\eta}$. By Theorems 3, 1(a),

$$\begin{aligned} \int_E \frac{|f_n - f|}{1 + |f_n - f|} d\bar{\mu} &\leq \int_{E'_{n\eta}} \frac{|f_n - f|}{1 + |f_n - f|} d\bar{\mu} + \int_{E - E'_{n\eta}} \frac{|f_n - f|}{1 + |f_n - f|} d\bar{\mu} \\ &\leq \bar{m}E'_{n\eta} + \eta\bar{m}(E - E'_{n\eta}), \end{aligned}$$

which approaches zero since η is arbitrary.

But owing to the non-additivity of the integral (see remark 4), the converse is not necessarily true as is shown by the following example.

Let $E_1', E_2', \dots$ be the denumerable non-measurable Vitali sets such that $E = (0, 1) = E_1' + E_2' + \dots$, $E_i' E_j' = 0$ and $\bar{m}E_i' = \bar{m}E_j' = q > 0$ for $i \neq j$ [4, p. 575; 5, p. 401]. Define $E_n = E_1' + E_2' + \dots + E_n'$ and define $f_n = 0$ on E_n and 1 on $E - E_n$, then it is obvious that $f_n \rightarrow 0 \equiv f$ on E . Now

$$\begin{aligned} \int_E f_n d\bar{\mu} &= \bar{m}E - \bar{m}E_n \rightarrow 0, \text{ while for } 0 < \eta < 1, \bar{m}E_{n\eta} = \bar{m}E(|f_n| > \eta) = \\ &= \bar{m}(E - E_n) \geq \bar{m}E'_{n+1} = q > 0. \end{aligned}$$

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